Strong convergence properties of the Ninomiya-Victoir scheme and applications to multilevel Monte Carlo

Benjamin Jourdain

CERMICS
Université Pais Est

Conference Vlad Bally, le Mans, 8 october 2015
Joint work with Anis Al Gerbi and Emmanuelle Clément

Outline

1. The Ninomiya-Victoir scheme
2. Strong convergence properties
   - Interpolation and strong convergence
   - Commutation of the Brownian vector fields
3. Multilevel Monte Carlo
   - Antithetic Monte Carlo Multilevel (AMLMC)

Stochastic Differential Equation

We are interested in the simulation of the Itô-type SDE

$$\begin{cases} dX_t = b(X_t)dt + \sum_{j=1}^{d} \sigma^j(X_t)dW_t^j \\ X_0 = x_0 \end{cases}$$

Where:

- $x_0 \in \mathbb{R}^n$.
- $(X_t)_{t\in[0,T]}$ is a $n-$dimensional stochastic process.
- $W = \left(W^1, \ldots, W^d\right)$ is a $d-$dimensional standard Brownian motion.
- $b, \sigma^1, \ldots, \sigma^d : \mathbb{R}^n \to \mathbb{R}^n$ Lipschitz with $\sigma^1, \ldots, \sigma^d$ $\mathcal{C}^1$.

This stochastic differential equation can be written in Stratonovich form:

$$\begin{cases} dX_t = \sigma^0(X_t)dt + \sum_{j=1}^{d} \sigma^j(X_t) \circ dW_t^j \\ X_0 = x_0 \end{cases}$$

where $\sigma^0 = b - \frac{1}{2} \sum_{j=1}^{d} \partial\sigma^j \sigma^j$ and $\partial\sigma^j$ is the Jacobian matrix of σ^j.

Related Ordinary Differential Equations

For $j \in \{0, \dots, d\}$ and $x \in \mathbb{R}^n$, let $(\exp(t\sigma^j)x)_{t\in\mathbb{R}}$ solve the ODE

$$\begin{cases} \frac{d\exp(t\sigma^j)x}{dt} = \sigma^j\left(\exp(t\sigma^j)x\right) \\ \exp(0\sigma^j)x = x \end{cases}$$

One has $\frac{d^2\exp(t\sigma^j)x}{dt^2} = \partial\sigma^j\sigma^j\left(\exp(t\sigma^j)x\right)$ so that by Itô's formula, for $j \in \{1, \dots, d\}$,

$$\begin{aligned} d\exp(W_t^j\sigma^j)x &= \sigma^j\left(\exp(W_t^j\sigma^j)x\right)dW_t^j + \frac{1}{2}\partial\sigma^j\sigma^j\left(\exp(W_t^j\sigma^j)x\right)dt \\ &= \sigma^j\left(\exp(W_t^j\sigma^j)x\right)\circ dW_t^j \end{aligned}$$

Commutative case

Assume that

$$\forall j, m \in \{0, \dots, d\},\ \partial\sigma^m \sigma^j = \partial\sigma^j \sigma^m \text{ i.e. } [\sigma^m, \sigma^j] = 0 \qquad (1)$$

By Frobenius theorem, $\exists \varphi : \mathbb{R}^{d+1} \to \mathbb{R}^n$ such that

$$\begin{cases} \varphi(0, \dots, 0) = x_0 \\ \forall j \in \{0, \dots, d\},\ \frac{\partial \varphi}{\partial s_j}(s_0, s_1, \dots, s_d) = \sigma^j\left(\varphi(s_0, s_1, \dots, s_d)\right). \end{cases}$$

(1) $\Leftrightarrow$ Schwarz compatibility between $\frac{\partial^2 \varphi}{\partial s_j \partial s_m}$ and $\frac{\partial^2 \varphi}{\partial s_m \partial s_j}$.

$$\text{Then } (X_t)_{t \geq 0} = (\varphi(t, W_t^1, \dots, W_t^d))_{t \geq 0}.$$

Ninomiya-Victoir scheme

Let $N \in \mathbb{N}^*$, $(t_k = \frac{kT}{N})_{0 \le k \le N}$, $\Delta W_{t_{k+1}} = W_{t_{k+1}} - W_{t_k}$ and $\eta = (\eta_k)_{1 \le k \le N}$ be a sequence of i.i.d. Rademacher random variables independent of W such that $\mathbb{P}(\eta_k = 1) = \mathbb{P}(\eta_k = -1) = \frac{1}{2}$.

Scheme

Starting point: $X_{t_0}^{NV,\eta} = x_0$. For $k \in \{0 \dots, N-1\}$:
If $\eta_{k+1} = 1$:

$$X_{t_{k+1}}^{NV,\eta} = \exp\left(\frac{t_1}{2}\sigma^0\right) \exp\left(\Delta W_{t_{k+1}}^d \sigma^d\right) \dots \exp\left(\Delta W_{t_{k+1}}^1 \sigma^1\right) \exp\left(\frac{t_1}{2}\sigma^0\right) X_{t_k}^{NV,\eta}$$

And if $\eta_{k+1} = -1$:

$$X_{t_{k+1}}^{NV,\eta} = \exp\left(\frac{t_1}{2}\sigma^0\right) \exp\left(\Delta W_{t_{k+1}}^1 \sigma^1\right) \dots \exp\left(\Delta W_{t_{k+1}}^d \sigma^d\right) \exp\left(\frac{t_1}{2}\sigma^0\right) X_{t_k}^{NV,\eta}$$

Under commutation (1), by induction, $\forall k \in \{0, \dots, N\}$,
$X_{t_k}^{NV,\eta} = X_{t_k}^{NV,-\eta} = \varphi(t_k, W_{t_k}^1, \dots, W_{t_k}^d) = X_{t_k}$.

Order 2 of weak convergence

Denoting by $(X_t^x)_{t\geq 0}$ the solution to the SDE starting from $X_0^x = x \in \mathbb{R}^n$, for $f : \mathbb{R}^n \to \mathbb{R}^n$ smooth, $u(t,x) := \mathbb{E}\left[f(X_t^x)\right]$ solves the Feynman-Kac PDE

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = Lu(t,x), \ (t,x) \in [0,\infty) \times \mathbb{R}^n \\ u(0,x) = f(x), \ x \in \mathbb{R}^n \end{cases}$$

with $L := b.\nabla_x + \frac{1}{2}\mathrm{Tr}\left[(\sigma^1,\ldots,\sigma^d)(\sigma^1,\ldots,\sigma^d)^*\nabla_x^2\right] = \sigma^0 + \frac{1}{2}\sum_{j=1}^d (\sigma^j)^2$ the infinitesimal generator.

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t} Lu = L\frac{\partial}{\partial t} u = L^2 u$$

$$\text{and } u(t_1,x) = f(x) + t_1 Lf(x) + \frac{t_1^2}{2} L^2 f(x) + \mathcal{O}(t_1^3)$$

Ninomiya and Victoir have designed their scheme so that

$$\mathbb{E}[f(X_{t_1}^{NV,\eta})] = f(x_0) + t_1 Lf(x_0) + \frac{t_1^2}{2} L^2 f(x_0) + \mathcal{O}(t_1^3).$$

One step error $\mathcal{O}(\frac{1}{N^3}) \stackrel{N\text{steps}}{\longrightarrow} \mathcal{O}(\frac{1}{N^2})$ global error.

Convergence in total variation results

Replace $W^j_{t_{k+1}} - W^j_{t_k}$ by $\sqrt{T/N} Z^j_{k+1}$ where the random variables $(Z^j_k)_{1 \leq j \leq d, k \geq 1}$ are independent and such that

- $\mathbb{E}[Z^j_k] = \mathbb{E}[(Z^j_k)^3] = \mathbb{E}[(Z^j_k)^5] = 0$, $\mathbb{E}[(Z^j_k)^2] = 1$, $\mathbb{E}[(Z^j_k)^4] = 3$,
- $\exists$ a non-empty open ball B and $\varepsilon > 0$ such that $\mathcal{L}(Z^j_k) >> \varepsilon 1_B(x)dx$.

Theorem (Bally Rey 15)

Assume that $\forall j \in \{0, \dots, d\}$, $\sigma_j : \mathbb{R}^n \to \mathbb{R}^n$ *is* $\mathcal{C}^\infty$ *bounded together with its derivatives. Then* $\exists C \in (0, \infty)$, $\forall f \in \mathcal{C}^6_b(\mathbb{R}^n)$,

$$\forall N, \sup_{0 \leq k \leq N} |\mathbb{E}[f(X_{\frac{kT}{N}})] - \mathbb{E}[f(X^{NV,\eta}_{\frac{kT}{N}})]| \leq C \frac{\sup_{\alpha \in \mathbb{N}^n : |\alpha| \leq 6} \|\partial^\alpha f\|_\infty}{N^2}.$$

If moreover uniform ellipticity holds, then
$\forall 0 < S \leq T, \exists C(S) \in (0, \infty)$, $\forall f : \mathbb{R}^n \to \mathbb{R}^n$ *measurable and bounded,*

$$\forall N, \sup_{k : \frac{kT}{N} \geq S} |\mathbb{E}[f(X_{\frac{kT}{N}})] - \mathbb{E}[f(X^{NV,\eta}_{\frac{kT}{N}})]| \leq \frac{C(S) \|f\|_\infty}{N^2}.$$

Motivation for studying strong convergence

Derivation of a rate of convergence: *Bayer Fritz 13* obtain convergence in $\alpha < \frac{1}{2}$-Hölder norm by rough paths theory but with no rate.

Multilevel Monte Carlo estimator of $\mathbb{E}[f(X_T)]$

$$\frac{1}{M_0}\sum_{i=1}^{M_0} f\left(X_T^{2^0,i,0}\right) + \sum_{l=1}^{L}\frac{1}{M_l}\sum_{i=1}^{M_l}\left(f\left(X_T^{2^l,i,l}\right) - f\left(X_T^{2^{l-1},i,l}\right)\right)$$

Debrabant Rössler 15 replace $X^{2^L,i,l}$ by a scheme with high order of weak convergence to reduce the bias $\rightarrow$ variance controlled by strong error.

Giles Szpruch 14 replace $f(X_T^{2^l,i,l}) - f(X_T^{2^{l-1},i,l})$ by $\frac{f(X_T^{2^l,i,l})+f(\tilde{X}_T^{2^l,i,l})}{2} - f(X_T^{2^{l-1},i,l})$ with $\tilde{X}^{2^l,i,l}$ antithetic version of $X^{2^l,i,l}$ to achieve

$$\mathrm{Var}\left[\frac{f(X_T^{2^l,i,l})+f(\tilde{X}_T^{2^l,i,l})}{2} - f(X_T^{2^{l-1},i,l})\right] \leq \frac{C}{2^{2l}}.$$

$\longrightarrow$ complexity $\mathcal{O}(\epsilon^{-2})$ for the precision ϵ.

Outline

Interpolation between the grid points t_k

Natural interpolation between $X^{NV,\eta}_{t_k}$ and $X^{NV,\eta}_{t_{k+1}}$ given for $t \in [t_k, t_{k+1}]$ by

$$1_{\{\eta_{k+1}=1\}} \exp(\frac{\Delta t}{2}\sigma_0) \exp(\Delta W^d_t \sigma^d) \dots \exp(\Delta W^1_t \sigma^1) \exp(\frac{\Delta t}{2}\sigma_0) X^{NV,\eta}_{t_k}$$

$$+ 1_{\{\eta_{k+1}=-1\}} \exp(\frac{\Delta t}{2}\sigma_0) \exp(\Delta W^1_t \sigma^1) \dots \exp(\Delta W^d_t \sigma^d) \exp(\frac{\Delta t}{2}\sigma_0) X^{NV,\eta}_{t_k}$$

where $(\Delta t, \Delta W_t) = (t - t_k, W_t - W_{t_k}) \longrightarrow$ very complicated Itô decomposition involving the flows of the ODEs. We rather set

$$X^{NV,\eta}_t = X^{NV,\eta}_{t_k} + \sum_{j=1}^{d} \int_{t_k}^{t} \sigma^j \left(\bar{X}^{j,\eta}_s\right) \circ dW^j_s + \frac{1}{2} \int_{t_k}^{t} \sigma^0 \left(\bar{X}^{0,\eta}_s\right) + \sigma^0 \left(\bar{X}^{d+1,\eta}_s\right) ds$$

where for $s \in]t_k, t_{k+1}]$, if $\eta_{k+1} = 1$, $\bar{X}^{0,\eta}_s = \exp\left(\frac{\Delta s}{2}\sigma^0\right) X^{NV,\eta}_{t_k}$,

$$\text{for } j \in \{1, \dots, d\}, \; \bar{X}^{j,\eta}_s = \exp\left(\Delta W^j_s \sigma^j\right) \bar{X}^{j-1,\eta}_{t_{k+1}}$$

and $\bar{X}^{d+1,\eta}_s = \exp\left(\frac{\Delta s}{2}\sigma^0\right) \bar{X}^{d,\eta}_{t_{k+1}}$ and $\bar{X}^{j,\eta}$ is defined symmetrically by backward induction on j when $\eta_{k+1} = -1$.

Order 1/2 of strong convergence

Theorem (Strong convergence)

Assume that

- $b : \mathbb{R}^n \to \mathbb{R}^n$ *is Lipschitz*
- $\forall j \in \{1, \ldots, d\}$, $\sigma_j : \mathbb{R}^n \to \mathbb{R}^n$ *is* $\mathcal{C}^1$ *with a bounded Jacobian matrix* $\partial\sigma_j$ *and such that* $\partial\sigma_j\sigma_j$ *is Lipschitz.*

Then $\forall p \in [1, +\infty)$,

$$\exists C_{NV} < \infty, \forall N \in \mathbb{N}^*, \ \mathbb{E}\left[\sup_{t \leq T} \left\| X_t - X_t^{NV,\eta} \right\|^{2p} \middle| \eta \right]^{1/(2p)} \leq \frac{C_{NV}\left(1 + \|x_0\|\right)}{\sqrt{N}}$$

Stable convergence of the normalized error

Theorem (Stable convergence)

Assume that

- σ^0 *is* $\mathcal{C}^2$, *Lipschitz and with polynomialy growing* 2^{nd} *order deriv.,*
- $\forall j \in \{1, \ldots, d\}$, σ^j *is* $\mathcal{C}^3$, *Lipschitz,* $\partial\sigma_j$ *is Lipschitz and the derivatives of* $\partial\sigma_j\sigma_j$ *have polynomial growth,*
- $\forall j, m \in \{1, \ldots, d\}\,, \partial\sigma^j\sigma^m$ *is Lipschitz.*

Then, as $N \to \infty$, $(\sqrt{N}(X_t^{NV,\eta} - X_t))_{0\le t\le T}$ *converge in law stably towards the unique solution* $(V_t)_{0\le t\le T}$ *to the affine equation:*

$$V_t = \sqrt{T/2}\sum_{j=1}^{d}\sum_{m=1}^{j-1}\int_0^t \left(\partial\sigma^j\sigma^m - \partial\sigma^m\sigma^j\right)(X_s)\,dB_s^{j,m}$$
$$+\int_0^t \partial b\,(X_s)\,V_s ds + \sum_{j=1}^{d}\int_0^t \partial\sigma^j\,(X_s)\,V_s dW_s^j,$$

where B *is a* $d(d-1)/2$*-dimensional Brownian motion indep. of* W*.*

Stable convergence

Remark

- The limit does not depend on η.
- If the Brownian vector fields commute, i.e. $\forall j, m \in \{1, \dots, d\}$, $\partial\sigma_j\sigma_m = \partial\sigma_m\sigma_j$, then the limit is 0.

Outline

The commutative case

Assumption

We assume that $\forall j, m \in \{1, \ldots, d\}, \partial\sigma^j\sigma^m = \partial\sigma^m\sigma^j$

The order of integration of these Brownian vector fields no longer matters and η is useless.
We use the natural interpolation given for $t \in [t_k, t_{k+1}]$ by

$$X_t^{NV,\eta} = \exp(\frac{\Delta t}{2}\sigma_0)\exp(\Delta W_t^d\sigma^d)\ldots\exp(\Delta W_t^1\sigma^1)\exp(\frac{\Delta t}{2}\sigma_0)X_{t_k}^{NV,\eta},$$

where $\Delta t = t - t_k$ and $\Delta W_t = W_t - W_{t_k}$.

Order one of strong convergence

Theorem (Strong convergence)

We assume that

- $b, \sigma^0, \sigma^1, \ldots, \sigma^d$ *are Lipschitz,*
- $\forall j \in \{1, \ldots, d\}$, σ^j *is* $\mathcal{C}^1$,
- σ^0 *is* $\mathcal{C}^2$ *with second order derivatives growing polynomially,*
- $\forall j, m \in \{1, \ldots, d\}, \partial\sigma^j\sigma^m = \partial\sigma^m\sigma^j$ *i.e.* $[\sigma^j, \sigma^m] = 0$.

Then

$$\exists C_{NV} < \infty, \forall N \in \mathbb{N}^*, \ \mathbb{E}\left[\sup_{t \le T} \left\| X_t - X_t^{NV} \right\|^{2p}\right]^{1/(2p)} \le \frac{C_{NV}(1 + \|x_0\|)}{N}$$

Under the commutativity of the Brownian vector fields, it is possible to implement the Milstein scheme which also exhibits order one of strong convergence.

Stable convergence of the normalized error

Theorem (Stable convergence)

We assume that

- $\forall j \in \{0, \ldots, d\}$, σ^j *is* $\mathcal{C}^3$ *with bounded derivatives,*
- $\forall j, m \in \{1, \ldots, d\}, \partial\sigma^j\sigma^m = \partial\sigma^m\sigma^j$ *i.e.* $[\sigma^j, \sigma^m] = 0$.

Then $(N(X_t^{NV} - X_t))_{0 \le t \le T}$ *converge in law stably towards the unique solution* $(V_t)_{0 \le t \le T}$ *to the following affine equation*

$$V_t = \frac{T}{2\sqrt{3}} \sum_{j=1}^{d} \int_0^t \left(\partial\sigma^0\sigma^j - \partial\sigma^j\sigma^0\right)(X_s)\, dB_s^j$$

$$+ \int_0^t \partial b\,(X_s)\, V_s ds + \sum_{j=1}^{d} \int_0^t \partial\sigma^j\,(X_s)\, V_s dW_s^j$$

with B *a standard* d*-dimensional Brownian motion independent of* W.

The limit vanishes when all the vector fields $\sigma^0, \sigma^1, \ldots, \sigma^d$ commute.

Outline

The Giles-Szpruch scheme

$$\begin{cases} X_0^{GS} = x_0 \text{ and for } k \in \{0,\ldots,N-1\}, \\ X_{t_{k+1}}^{GS} = X_{t_k}^{GS} + b\left(X_{t_k}^{GS}\right)(t_{k+1}-t_k) + \sum_{j=1}^{d} \sigma^j\left(X_{t_k}^{GS}\right)\left(W_{t_{k+1}}^j - W_{t_k}^j\right) \\ +\frac{1}{2}\sum_{j,m=1}^{d} \partial\sigma^j\sigma^m\left(X_{t_k}^{GS}\right)\left((W_{t_{k+1}}^j - W_{t_k}^j)(W_{t_{k+1}}^m - W_{t_k}^m) - 1_{\{j=m\}}(t_{k+1}-t_k)\right) \end{cases}$$

Strong coupling (Giles Szpruch 15)

- $X^{GS,N}$ scheme with N steps
- $X^{GS,2N}$ scheme with $2N$ steps
- $\tilde{X}^{GS,2N}$ scheme with $2N$ steps and intervertion of the increments $(W_{\frac{k+1/2}{N}} - W_{\frac{k}{N}})$ and $(W_{\frac{k+1}{N}} - W_{\frac{k+1/2}{N}})$ for all $k \in \{0,\ldots,N-1\}$.

Assume that $b, \sigma^1, \ldots, \sigma^d$ $\mathcal{C}^2$ with bounded derivatives. Then,

$$\exists C < \infty, \forall N \in \mathbb{N}^*, \mathbb{E}\left[\left\|\frac{1}{2}\left(\tilde{X}_T^{GS,2N} + X_T^{GS,2N}\right) - X_T^{GS,N}\right\|^{2p}\right]^{1/(2p)} \leq \frac{C}{N}$$

Coupling with the Ninomiya-Victoir scheme

Theorem (Strong convergence)

Assume that

- $\forall j \in \{1, \dots, d\}, \sigma^j$ *is* $\mathcal{C}^3$ *with bounded first and second order derivatives and with polynomially growing third order derivatives,*
- $\forall j, m \in \{1, \dots, d\}$, $\partial\sigma_j\sigma_m$ *is Lipschitz,*
- b *is* $\mathcal{C}^2$ *with bounded derivatives,*

Then, $\forall p \geq 1, \ \exists C < \infty, \ \forall N \in \mathbb{N}^*$,

$$\mathbb{E}\left[\left\|\frac{1}{2}(X_T^{NV,\eta,N} + X_T^{NV,-\eta,N}) - X_T^{GS,N}\right\|^{2p} \Bigg| \eta\right]^{1/(2p)} \leq \frac{C}{N}.$$

Derived multilevel estimators

Strong coupling with order one between successive levels $\longrightarrow$ Optimal complexity $\mathcal{O}(\epsilon^{-2})$ where ϵ is the root mean-square error (accuracy).

antithetic NV-GS

$$\frac{1}{M_0}\sum_{i=1}^{M_0} f(X_T^{GS,2^0,i}) + \sum_{l=1}^{L-1}\frac{1}{M_l}\sum_{i=1}^{M_l}\left(\bar{f}_2(X_T^{GS,2^l,i}) - f(X_T^{GS,2^{l-1},i})\right)$$

$$+\frac{1}{M_L}\sum_{i=1}^{M_L}\left(\bar{f}_4(X_T^{NV,2^l,i}) - f(X_T^{GS,2^{L-1},i,l})\right) \text{ where}$$

$$\bar{f}_2(X_T^{GS,2^l}) = \frac{1}{2}(f(X_T^{GS,2^l}) + f(\tilde{X}_T^{GS,2^l}))$$

$$\bar{f}_4(X_T^{NV,2^l}) = \frac{1}{4}(f(X_T^{NV,\eta,2^l}) + f(X_T^{NV,-\eta,2^l}) + f(\tilde{X}_T^{NV,\eta,2^l}) + f(\tilde{X}_T^{NV,-\eta,2^l}))$$

Derived multilevel estimators

antithetic NV

$$\frac{1}{M_0}\sum_{i=1}^{M_0}\bar{f}_2(X_T^{NV,2^0,i}) + \sum_{l=1}^{L}\frac{1}{M_l}\sum_{i=1}^{M_l}\left(\bar{f}_4(X_T^{NV,2^l,i}) - \bar{f}_2(X_T^{NV,2^{l-1},i})\right)$$

where

$$\bar{f}_2(X_T^{NV,2^l}) = \frac{1}{2}(f(X_T^{NV,\eta,2^l}) + f(X_T^{NV,-\eta,2^l}))$$

$$\bar{f}_4(X_T^{NV,2^l}) = \frac{1}{4}(f(X_T^{NV,\eta,2^l}) + f(X_T^{NV,-\eta,2^l}) + f(\tilde{X}_T^{NV,\eta,2^l}) + f(\tilde{X}_T^{NV,-\eta,2^l}))$$

ClarkCameron SDE

$$\begin{cases} dX_t^1 = dW_t^1 \\ dX_t^2 = X_t^1 dW_t^2 \end{cases}$$

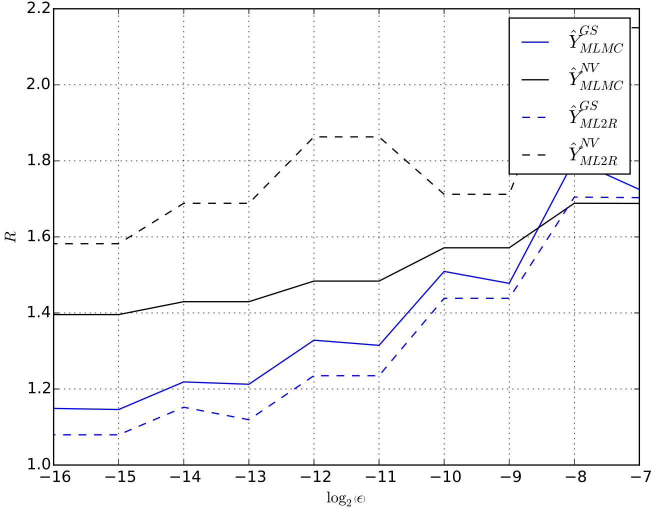
Parameters

- $X_0^1 = X_0^2 = 0.$
- $\mu = T = 1$

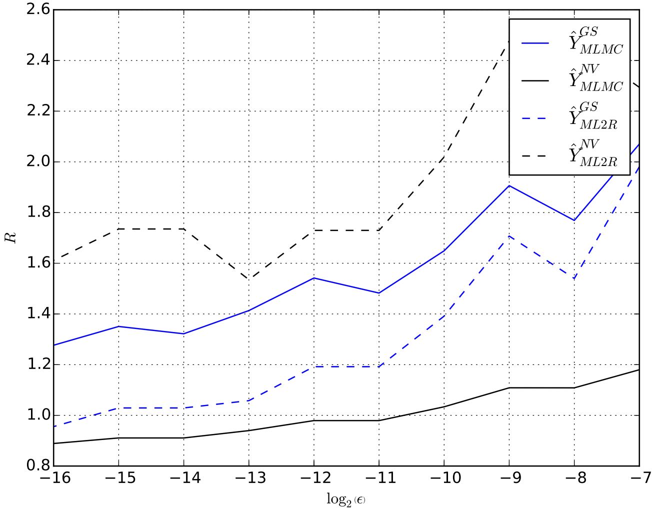
$$\sigma_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 \\ x_1 \end{pmatrix}, \ \partial\sigma_1 = 0, \ \partial\sigma_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\partial\sigma_1\sigma_2 = 0 \neq \partial\sigma_2\sigma_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$f(x_1, x_2) = \cos(x_2)$, $R = \dfrac{\text{Computation time of } \hat{Y}}{\text{Computation time of } \hat{Y}_{MLMC}^{NV-GS}}$



$f(x_1, x_2) = x_2^+, \; R = \frac{\text{Computation time of } \hat{Y}}{\text{Computation time of } \hat{Y}^{NV-GS}_{MLMC}}$



Heston model

$$\begin{cases} dX_t^1 = (r - \frac{X_t^2}{2})dt + \sqrt{X_t^2}dW_t^1 \\ dX_t^2 = \kappa(\theta - X_t^2)dt + \sigma\sqrt{X_t^2}dW_t^2 \end{cases}$$

Parameters

- $X_0^1 = 0$, $X_0^2 = 1$,
- $T = 1$, $\kappa = 0.5$, $\theta = 0.9$, $\sigma = 0.05$

$f(x_1, x_2) = e^{-rT}(e^{x_1} - 1)^+$, $R = \frac{\text{Computation time of } \hat{Y}}{\text{Comp. time of } \hat{Y}^{NV-GS}_{MLMC}}$